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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Results of a simulation study of the application of the complex one dimensional maximum entropy spectral analysis algorithm to the problem of localizing and resolving discrete targets using a linear array are reported. Of particular interest in this study were the errors in bearing estimation and statistical stability of the estimated array response.

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SUMMARY

The rms error in the source bearing estimates was observed to in-
crease with decreasing signal-to-noise ratio (SNR), when the model
order (p) approached the number of array elements (N = 31), and when
the source of interest was not well separated from adjacent sources.
Typical rms errors observed for isolated sources (for $p \simeq N/3$) ranged
from 0.02 λ/D at a signal-to-noise ratio (SNR) of 17 db to 0.5 λ/D at
-9 dB. The statistical variability and number of spurious peaks in the
array response were observed to increase rapidly as p approached N. At
an SNR of less than 0 dB the array response, even when averaged over 50
independent data sets, indicated numerous spurious sources when the
model order was large. The equivalent number of degrees of freedom (df)
of the array response, estimated as twice the square of the average
divided by the variance of the response, were calculated and compared
with df = N/p as conjectured by Parzen (1970) and demonstrated (asymp-
totically) by Kromer (1969). Parzen's conjecture was consistent with
the observed results for $p \ll N$, but was found to be very optimistic
at high model orders. It was also found that the spatial response to an
individual point source increases in width in the presence of nearby tar-
gets as well as with decreasing signal-to-noise ratio; and while the
spatial response to a single point source is typically much narrower for
maximum entropy processing, the minimum angular separation required for
the resolution of adjacent sources is only slightly less (roughly a fac-
tor of one-half) than that of a conventional beamformer even at high
signal-to-noise ratios.

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MOTIVATION

Consider the problem of estimating the bearing and strength of a single point source using a linear array where the source may either be actively generating its own signal or reflecting a transmitted waveform. Assuming the source and receiver are separated sufficiently to permit approximating the received signal as a plane wave, the fields in the vicinity of the array may be written as

$$s(\bar{x},t) = p(t-\bar{\alpha}\cdot\bar{x}) \tag{1}$$

where $\bar{\alpha} = \bar{v}/|v|^2$, $\bar{v}$ is the velocity vector of the plane wave, and $\bar{x}$ specifies the field point. Further, if the signal is assumed to be narrowband, then

$$s(\bar{x},t) = w(t-\bar{\alpha}\cdot\bar{x})e^{-i(\omega_0\bar{\alpha}\cdot\bar{x}-\omega_0 t)} \tag{2}$$

where $w(t-\bar{\alpha}\cdot\bar{x})e^{-i\omega_0\bar{\alpha}\cdot\bar{x}}$ is the complex envelope of the waveform observed at the field point $\bar{x}$. The array geometry is shown in Figure 1. Assuming the linear array consists of N elements spaced Δx apart and lies along the x axis of the coordinate system with the origin at one end, then the output of the nth element is

$$z_n(t) = s(n\Delta x\ \hat{x},t)$$

$$\simeq w(t - n\Delta x\ \alpha_x)e^{-i2\pi k_x n\Delta x}e^{i\omega_0 t} \tag{3}$$

where $\hat{x}$ is a unit vector along the x-axis,

$$k_x = \omega_0\alpha_x = \frac{1}{\lambda}\sin(\theta)$$

is the x component of the vector wave number, θ is the angle between the direction of propagation, and the x axis, λ is the wavelength, and $\alpha_x = \alpha\cdot x = \alpha\sin(\theta)$ is the x component of the inverse velocity vector. Equation (3) may be approximated as

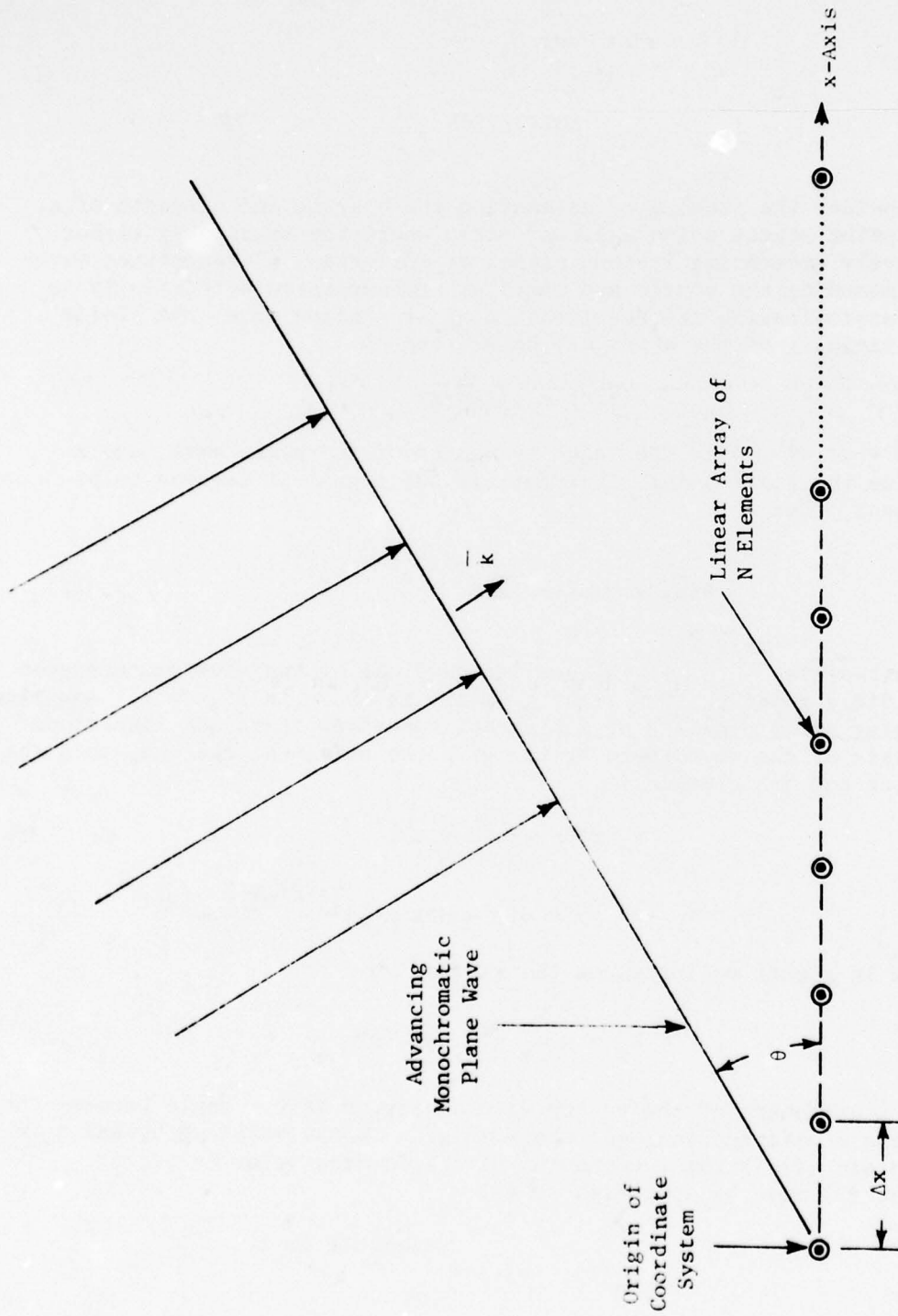$$z_n(t) \simeq w(t)\ e^{-i2\pi k_x n\Delta x}e^{i\omega_0 t} \tag{4}$$

FIGURE 1. THE ARRAY SYSTEM

if the transit time across the array is small compared to the time of variation of the signal envelope; i.e., the inverse of the signal bandwidth. Finally, the carrier term in Equation (4) may be removed by coherently detecting $z_n(t)$ to obtain

$$z_n'(t) = w(t)e^{-i2\pi k_x n\Delta x} . \tag{5}$$

All of the information relevant to the source strength and bearing is contained in the $z_n'(t)$ and their relative delay times[1]. The above equation indicates that, under the assumptions made, the fields measured along a linear array at time t vary sinusoidally as a function of x with frequency k_x cycles/unit distance. Knowledge of k_x and ω_o are sufficient to determine the direction of propagation in a nondispersive medium when the velocity of all possible plane waves are required to lie in the xy plane since k_y may be obtained from the dispersion relation

$$k_x^2 + k_y^2 = \frac{(2\pi)^2\omega_o^2}{c^2} \tag{6}$$

where c is the velocity of propagation. Thus, estimating the bearing to a single point target is nothing more than estimation of the frequency of the variation along the array. It is evident from the discussion above that the problem is simply one of spectral estimation. The conventional approach to this problem is beamforming which for the narrowband case reduces to calculation of the Discrete Fourier Transform of the array element outputs at a particular time. The search for high spatial resolution quite naturally leads to algorithms developed for high resolution spectrum analysis.

In an active sonar system the coherently detected array outputs are a superposition of terms of the form given in Equation (5), where w(t) is an appropriately delayed version of the transmitted signal envelope. The principal problem is to resolve scattering centers lying at roughly the same range (more precisely in the same range cell where a range cell is one-half the velocity of propagation times the inverse of the transmitted signal bandwidth) since scattering centers lying at differing ranges may be distinguished in the time domain.

In a passive system the problem is to resolve sources emitting signals which overlap in time. It is important to improve the

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[1] I. N. El-Behery and R. H. MacPhie, *Maximum-likelihood Estimation of Source Parameters from Time Sampled Outputs of a Linear Array*, J. Acoust. Soc. Am. 62, p. 125-134 (1977).

signal-to-noise (SNR) as much as possible by coherent integration prior to spectral (bearing) estimation as the maximum entropy method is quite sensitive to noise. In either case the appropriate coherent integration times are limited in the active case by transmitted signal bandwidth, and in the passive case by the frequency stability of the source. In many cases it is possible to improve the estimates still further by averaging the results of multiple independent observations obtained by appropriately sampling the coherently detected array outputs. This should be used whenever possible.

The principal interest was in active high resolution sonar systems for the purpose of localizing and resolving discrete targets where only a single observation of the coherently detected array outputs was available. In this report maximum entropy processing as a means of locating and resolving point sources based on a single time sample (observation) of the array outputs will be investigated. Details of the maximum entropy method will be discussed only briefly in the following section since these are readily found in the literature. The principal results of this study are presented in the SIMULATION AND RESULTS section. This section begins with a discussion of the choice of model order, and its relation to the statistical stability of the array response. This is followed by a discussion of the effects of SNR and model order on (1) errors in bearing estimation, (2) the width of the array response to individual sources, and (3) the minimum separation required for resolution of equal amplitude point sources. When possible our results are compared with those of other researchers. The report concludes with a summary of the principal findings.

MAXIMUM ENTROPY SPECTRAL ANALYSIS

The maximum entropy spectral analysis technique was first proposed by J. P. Burg in 1967 as a means of achieving high resolution spectral estimates[2]. This approach was later recognized to be identical to auto-regressive spectral analysis, linear predictive deconvolution filtering, and all pole modeling of the spectrum[3]. The entropy of a Gaussian band limited time series is proportional to

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[2] Burg, J. P., *Maximum Entropy Spectral Analysis*, presented at the 37th Annual International Meeting, Soc. of Explor. Geophys., Oklahoma City, Oklahoma (1967).

[3] R. N. McDonough, *Maximum Entropy Spatial Processing of Array Data*, Geophysics 39, p. 843-851 (1974).

$$H = \int_{-f_N}^{f_N} \log S(f) df \tag{7}$$

where $S(f)$ is the power spectrum, and f_N is the Nyquist frequency[4]. The power spectrum is the Fourier transform of the autocorrelation function of the time series which is known only over a finite interval. Burg suggested that the most reasonable choice for the unknown autocorrelations is the one which maximizes the entropy. Maximization of H subject to the constraints imposed by the known autocorrelations implies a power spectrum of the form

$$S(f) \frac{P_p \Delta t}{\left| 1 - \sum_{j=1}^{p} a_j^p e^{-i2\pi f j \Delta t} \right|^2} \tag{8}$$

where $1, a_1^p, a_2^p \ldots a_p^p$ are the coefficients of the $p+1$ length prediction error filter, and p_p is the prediction error[4]. Burg later suggested an approach by which the filter coefficients could be estimated directly from the data[5]. This approach is based on choosing the reflection coefficients (a_p^p) to minimize the prediction error power obtained from running the filter both forward and backward across the data and on the Levinson recursion algorithm, and is outlined in Reference 6. In this study the maximum entropy array response (spatial spectrum) was calculated using Equation (8) with the replacements $f \rightarrow \Delta x$, and $\Delta t \rightarrow \Delta k_x$ and using the coherently demodulated element outputs (at a specified time) as the sampled complex data series. Burg's method (modified for use with complex data) of coefficient calculation was used to obtain the results presented in this report because it is widely used, and has been reported to produce resolution superior to the Yule Walker

[4] Smylie, D. E., Clarke, G. K. C. and Ulrych, T. J., *Analysis of Irregularities in the Earth's Rotation,* in B. Alder, S. Fernback, and M. Rotenberg (Eds.) "Methods in Computational Physics, Vol. 13," New York: Academic Press, pp. 391-430 (1973).

[5] Burg, J. P., *A New Analysis Technique for Time Series Data,* presented at NATO Advanced Study Institute on Signal Processing, Enschede, Netherlands (1968).

[6] Andersen, N., *On the Calculation of Filter Coefficients for Maximum Entropy Spectral Analysis,* Geophysics, Vol. 39, No. 1, pp. 69-72, February 1974.

approach[7][8]. It has, however, been pointed out by Makhoul that the Burg method of calculating the reflection coefficients is only one of a number of "suboptimal" solutions to the lattice minimization problem[9]. The complex formulation of the maximum entropy algorithm is discussed in References 3 and 10.

SIMULATION AND RESULTS

In the following simulations a linear array has been assumed having 31 elements spaced one-half wavelength apart responding to inphase monochromatic signals generated by two point sources. The inphase condition was chosen because this is the most difficult condition to resolve. The signals observed by each element are assumed to be corrupted by narrowband Gaussian noise which is independent between elements. After coherent demodulation the output of the nth element is

$$z'_n(t) = \sum_{\ell=1}^{2} w^{\ell}(t)\, e^{-i2\pi k_x^{\ell} n\Delta x} + n_n(t) \tag{9}$$

where $n_n(t)$ is the envelope of the complex noise added to the nth element. The SNR's quoted in the remainder of this study are defined as the ratio of the signal power (generated by one of the sources) to noise power at the output of each element after demodulation.

At each combination of SNR and target separation 50 data sets (with independent noise realizations) corresponding to a single time sample of

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[3] ibid.

[7] T. J. Ulrych and T. N. Bishop, *Maximum Entropy Spectral Analysis and Autoregressive Decomposition*, Reviews of Geophysics and Space Physics 13, p. 183-200 (1975).

[8] H. R. Radoski, P. F. Fougere, and E. J. Zawalick, *A Comparison of Power Spectral Estimates and Applications of the Maximum Entropy Method*, J. Geophysical Research 80, p. 619-625 (1975).

[9] J. Makhoul, *Lattice Methods in Spectral Estimation*, in the "Proceedings of the RADC Spectrum Estimation Workshop," p. 159-173 (1978).

Texas Instruments, Inc., Technical Report No. 1, Advanced Signal Processing, ALEX(03)-TR-75-01, *The Maximum Entropy Spectrum and the Burg Technique*, by T. E. Barnard, 1975.

each of the N array elements were examined. Coefficients corresponding to model orders 1 through 30 were calculated for each data set. Spectra were calculated corresponding to model orders of 5, 10, 15, 20, 25, and 30. The reported angular separations were achieved by holding the bearing of one target fixed and moving the second target relative to the first. This was done so that changes in the response to one of the targets would be related to changes in target separation and not changes in target bearing. Only parameters of the array response related to the fixed target are reported. Some definite trends were observable in the results.

CHOICE OF MODEL ORDER

The estimation of the order of the appropriate maximum entropy model has been treated by a number of authors. Based on the results reported[7] by Ulrych and Bishop, Akaike's Final Prediction Error (FPE) Scheme was selected in which the optimum value of p is chosen as that value which minimizes the FPE given by

$$(FPE)_p = \frac{N + p}{N - p} P_p \quad .$$

Unfortunately, in this application the minimum of the FPE does not appear to indicate acceptable model orders. Typical results for the average and standard deviation of the FPE over 50 data sets for model orders 1 through 30 are shown in Figure 2. Two characteristics of the FPE are quite noticeable. First, the variance of the FPE increases with model order. This would appear to be related to the fact that the statistical stability of the estimated spectra decreases with increasing model order. And second, the FPE has a local minima at below 10, is relatively flat up to $p = 25$, and decreases rapidly for $p > 25$. Based on the average FPE, $p = 30$ would be chosen, however a comparison of the spectra obtained for differing model orders indicates that this is not acceptable since both the number of spurious peaks and statistical variability of the resulting spectra increase rapidly as p approaches N. Clearly if the calculated responses are to be meaningful both the number of spurious peaks and the statistical variability of the response must be controlled. This is particularly important when the responses are based on single observations of the array outputs and no averaging is performed. These problems are more acute at low SNR because the signal peaks are relatively smaller, but they can be significant even at high SNR when p is large. Even the average spectrum may be of marginal value when the model order approaches N. Figures 3a and 3b compare the average (over 50 data sets) spectrum at an SNR of 7 dB for $p = 30$, and $p = 15$; clearly the spectrum is useless for $p = 30$ due to the number of spurious peaks. At an SNR of -3 dB the largest model order producing a

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[7] ibid.

SNR = 7 dB
SEP = 2.0 λ/D

——— FPE
-------- Standard Deviation

Final Prediction Error

0.5

0.4

0.3

0.2

0.1
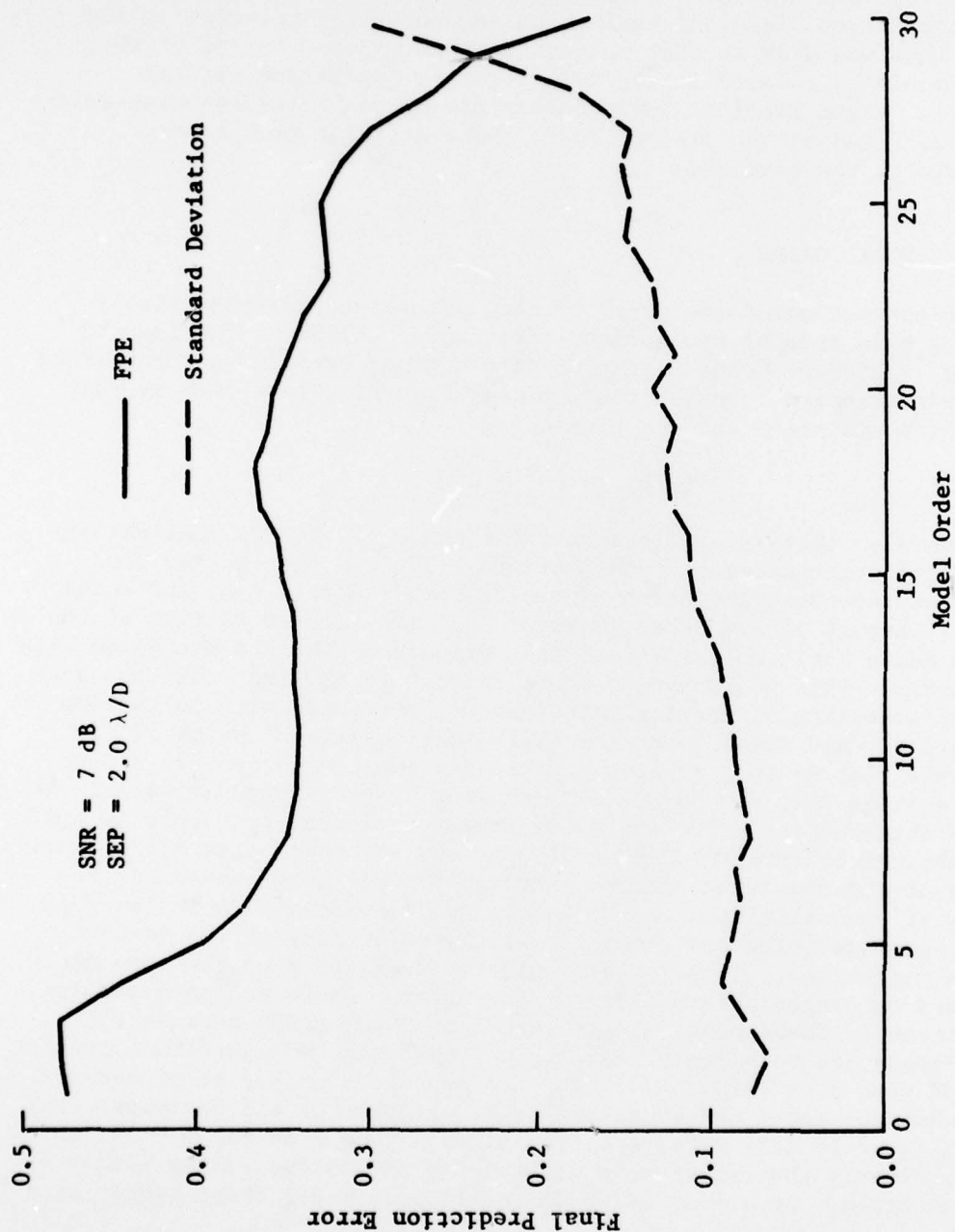
0.0

0   5   10   15   20   25   30
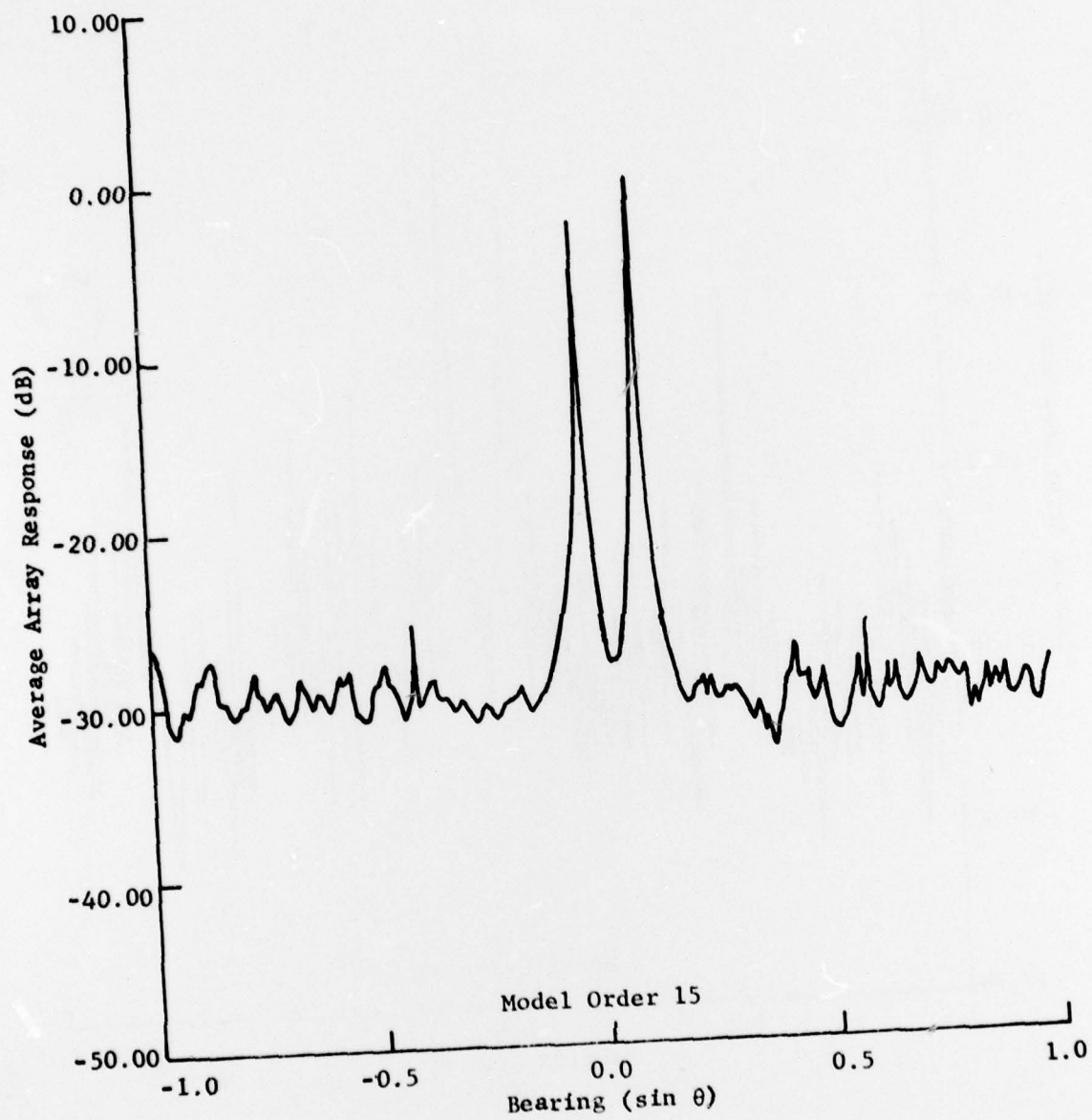
Model Order

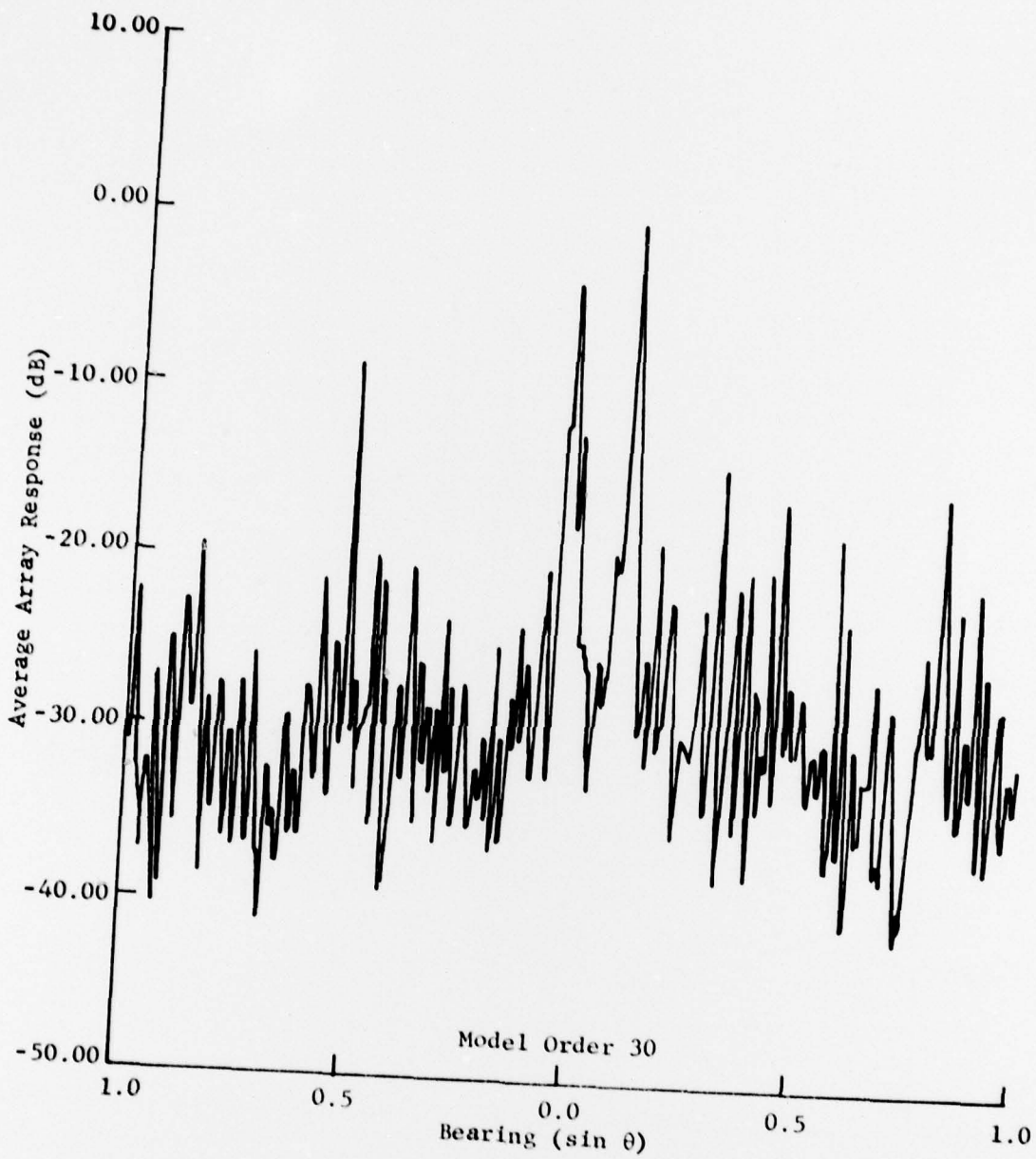FIGURE 2. FPE VERSUS MODEL ORDER

FIGURE 3a. AVERAGE ARRAY RESPONSE

FIGURE 3b.

useful average was p = 20. In any event, it appears that little is gained in terms of resolution or bearing estimation accuracy by choosing p > 10.

As in ordinary spectral estimations, the equivalent number of degrees of freedom of the array response may be estimated as

$$\nu = \left\{ \frac{2E^2(S(k_x))}{Var(S(k_x))} \right\} \quad \text{over all } k_x \qquad (10)$$

where $E(S(k_x))$ and $Var(S(k_x))$ are the mean and variance of the maximum entropy array response. This quantity provides a measure of the statistical stability of the response. Figure 4 illustrates the dependence of this quantity on the model order where the degrees of freedom were estimated from the mean and variance of the response over the 50 data sets for the SNR and target separation indicated. This behavior is typical of the results observed. It is apparent that the statistical variability of the array response increases rapidly as p approaches N. There are few theoretical results on the statistical properties of autoregressive spectral estimates for comparison with these results. It has, however, been conjectured by Parzen (1970)[11] and demonstrated by Kromer (1969)[12] that autoregressive modeled spectral estimates are asymptotically (for large p and N with p << N) distributed with ν = N/p degrees of freedom[13]. The statistical stability of the observed results is consistent with Parzen's conjecture when p << N, but is increasingly less stable as p approaches N as was expected from the asymptotic nature of the theoretical result.

It was concluded from observations that the best combination of statistical stability and source resolution was obtained for p ≈ 10 although p = 5 may be acceptable if the resolution requirements are not stringent. Choosing p = 1/3 the number of array elements appears to be a good starting point although the "optimum" choice depends on the SNR,

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[11] Parzen, E., *Multivariate Time Series Modeling*, in "Multivariate Time Series Analysis II," P. R. Kirshniah, Ed., Academic Press, New York, 1970.

[12] Kromer, R. E., *Asymptotic Properties of the Autoregressive Spectral Estimator*, Dissertation, Dept. of Statistics, Stanford University 1969.

[13] Gersch, W. and Liu, R. Z., *Autoregressive Model Spectral Estimation, Some Simulation Study Statistical Performance Results*, in "Proceedings of the RADC Spectrum Estimation Workshop," May 1978.
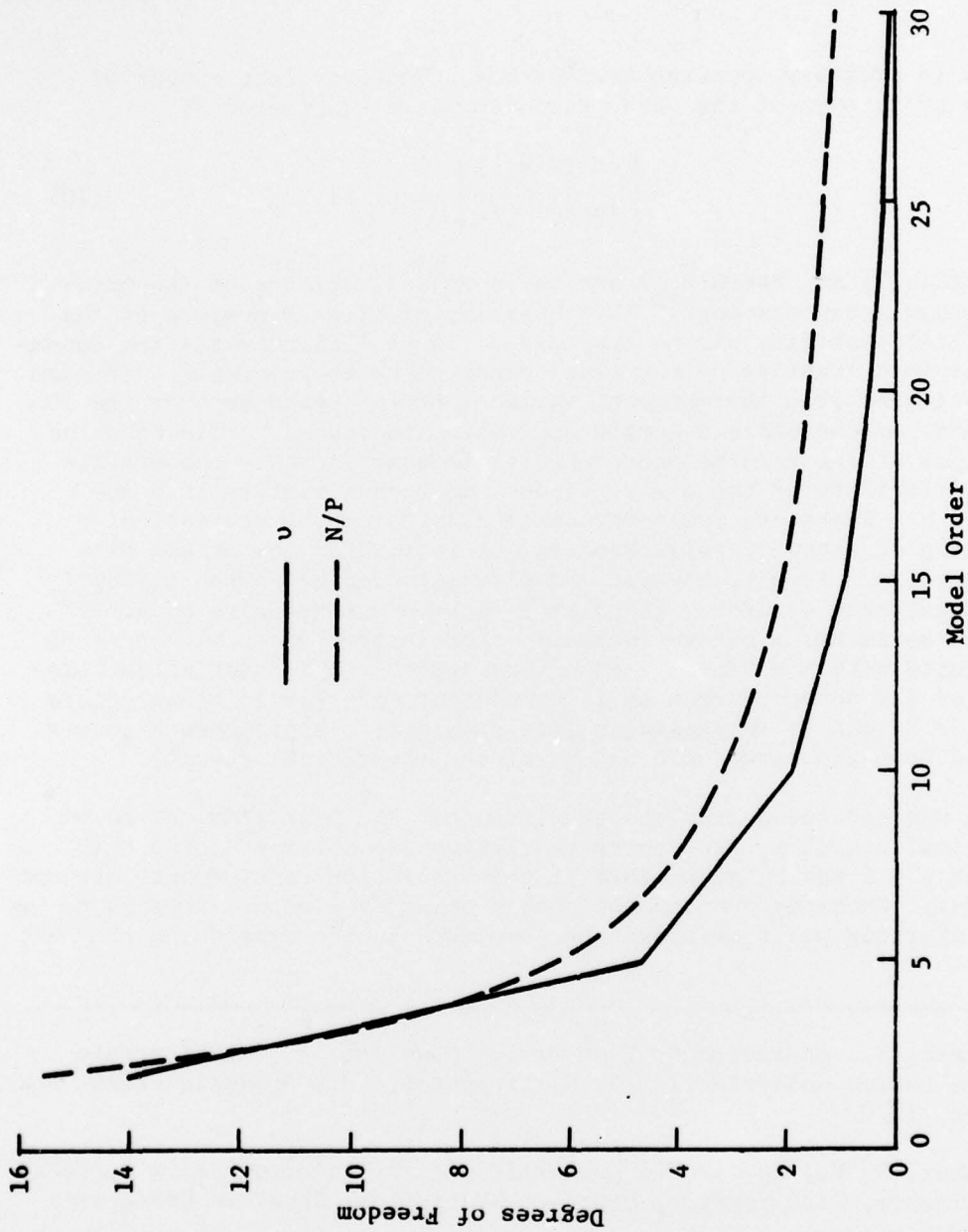
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FIGURE 4. DEGREES OF FREEDOM VS MODEL ORDER

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number of data records available, and the desired resolution, as well as the nature of the data. It is interesting to note that the choice of $p \simeq 10$ gives roughly the same stability as that obtained with conventional beamforming with an unweighted aperture (conventional spectral estimation with rectangular window).

BEARING ERROR

Source localization in azimuth is particularly important in many applications. It appears that the maximum entropy procedure provides excellent bearing estimates when the source of interest is well separated from other sources.

Source bearing was estimated by locating the largest peak in the vicinity of the known source location. Table 1 indicates the total rms

TABLE 1

RMS DEVIATION OF PEAK LOCATION FROM TRUE BEARING

| SNR (dB) | Separation* | RMS Deviation* Model Order | | |
		5	10	15
-9	∞**	0.7	0.5	0.5
	1.5	0.9	0.9	0.7
	2.0	1.0	0.9	0.6
-3	∞	0.2	0.2	0.2
	1.5	0.8	0.6	0.2
	2.0	1.2	0.2	0.2
7	∞	0.1	0.1	0.1
	1.5	0.9	0.2	0.2
	2.0	0.1	0.1	0.1
17	∞	0.02	0.02	0.02
	1.5	0.2	0.2	0.2
	2.0	0.05	0.05	0.04

*in multiples of λ/D
**single source

deviation of the estimated bearings from the true bearing. The total deviation error is a function of the model order, SNR, and target separation. It tends to increase with decreasing SNR when the model order is large and when the targets are not well resolved.

The total deviation error may be interpreted as consisting of two parts: the bias or average error (over the 50 data sets) in source location and deviations around this average. The bias is not significant when: only a single source is present, appears to be relatively unimportant for large separations, and tends to increase as the separation decreases. At high SNR the bias is the major source of error when multiple sources are present. The deviations around the average increase as the SNR decreases and is the major source of error for SNR < 0 dB (provided the sources are resolved). It is interesting to note from Table 1 that for isolated sources the errors for p = 5 are not significantly greater than for p = 10 or 15. If resolution of multiple sources is not required, p = 5 may be acceptable for source localization, and as noted previously gives a considerably more stable response.

It is instructive to compare the isolated target results with the Cramer-Rao lower bound on the rms error given (in multiples of λ/D) by[1]

$$\sigma = \frac{1}{\sqrt{Q}} \frac{\sqrt{1+N(SNR)}}{SNR} \frac{\sqrt{6}}{2\pi\sqrt{N^2-1}}$$

where Q is the number of independent time samples used in the estimate (in this case Q = 1). The C-R lower bound is the limit for the performance of any estimator. For SNR's of 17, 7, -3, and -9 dB the resulting C-R bounds are 1×10^{-2}, 3×10^{-2}, 1×10^{-1}, and 2×10^{-1}. The results in Table 1 for p = 10 are roughly twice the C-R bounds. This appears to compare reasonably well with the results on maximum liklihood estimation reported in Reference 1.

PEAK WIDTHS

Provided the targets were resolved, the 3 dB width of the response to the fixed source was always much less than the response width for conventional beamforming of $0.9 \lambda/D$. Even at SNR = 0 dB and p = 5 the average response width to a single source was only $0.5 \lambda/D$. The width increased as the SNR or model order decreased, and as target separation decreased.

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[1] ibid.

It is interesting to compare the observed single target response widths with those which can be calculated theoretically. The expression for the 3 dB response width for maximum entropy array processing may be obtained directly from the results obtained by Lacoss[14] in the time domain if the analogy between k_x and f is noted, and between Δx and Δt

$$\Delta k_x = \frac{2}{\pi \Delta x p^2 (SNR)} \tag{11}$$

where k_x is the 3-dB (wave number) width of the spatial response. This is easily related to the angular width $\Delta \theta$ since

$$k_x = \frac{1}{\lambda} \sin \theta$$

$$\Delta k_x = \frac{1}{\lambda} \cos \Delta \theta \tag{12}$$

$$\Delta \theta = \frac{2N}{\pi p^2 (SNR) \cos \theta} \left(\frac{\lambda}{D}\right) .$$

Table 2 compares the predicted and observed angular widths for the three SNR's when $\theta \simeq 0$ (source located on the perpendicular bisector of the array axis). Substitution of the appropriate model order predicts widths somewhat less than those actually observed. This is not surprising because Equation (11) was derived assuming exact knowledge of the signal autocorrelations rather than estimates.

TABLE 2

BEAMWIDTH COMPARISON

SNR (dB)	Observed*			Calculated
	p = 5	p = 10	p = 15	
17	8×10^{-3}	4×10^{-3}	3×10^{-3}	$3.9 \times 10^{-1}/p^2$
7	7×10^{-2}	3×10^{-2}	2×10^{-2}	$3.9 \times 10^{0}/p^2$
-3	9×10^{-1}	3×10^{-1}	5×10^{-2}	$3.9 \times 10^{1}/p^2$
-9	2×10^{0}	1×10^{0}	5×10^{-1}	$1.6 \times 10^{2}/p^2$

*in multiples of λ/D

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[14]Lacoss, R. T., *Data Adaptive Spectral Analysis Methods*, Geophysics, Vol. 36, No. 4, pp. 661-675 (August 1974).

It should be noted that while the resolution of a conventional beamformer is directly related to its response to a single point source, this is not the case for maximum entropy processing since the algorithm is nonlinear and superposition does not hold. It is sometimes stated that the maximum entropy spectrum has no sidelobes. This can be misleading since it applies only to an isolated source. There is, in fact, a nonlinear interaction between components as evidenced by the changes in response width as a function of target separation which leads to problems similar to those caused by sidelobes in conventional beamforming. Based on the observed single target response it might be concluded that the "resolution" was many orders of magnitude better than is actually the case.

Resolution is a difficult parameter to define for maximum entropy spectra. When the target separation is small, it is often difficult to determine if the observed peaks correspond to sources or are due to noise. For the purposes of this study equal amplitude targets were considered to be resolved when there was a dip in average (over 50 samples) response of 3 dB between peak responses to the targets. In general, the results were as follows: the minimum resolvable separation (MRS) increased as the SNR or model order decreased. The decrease in MRS with SNR may be partially compensated for by increasing the model order, however as noted previously, the number of spurious spectral peaks increased rapidly as the model order increased. For model order 10 the minimum resolvable separation (MRS) for SNR's of 17, 7, -3, and -9 dB were 0.7, 1.0, 1.5, and 2 λ/D. The MRS for model order 15 was only slightly better while the MRS for a model order of 5 was roughly twice that obtained with a model order of 10. In cases where the resolution requirements are not extreme a low model order may be acceptable.

For comparison the minimum angular separation required at a high SNR for the resolution of two equal amplitude in phase point targets using conventional (delay and sum) beamforming if 1.4 λ/D. An extensive study of the resolution of sinusoids is described by Marple (1976)[15]. The average MRS reported is roughly one-half that reported above[16]. While there are differences in the signals (real versus complex) and the methods employed, it appears that this discrepancy is primarily caused by the differing definition of resolution used in the current study.

[15] L. Marple *Conventional Fourier, Autoregressive, and Special ARMA Methods of Spectrum Analysis*, Engineers Dissertation, Department of Electrical Engineering, Stanford University (1976).